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A sequential parametric convex approximation method with applications to nonconvex truss topology design problems

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Abstract We describe a general scheme for solving nonconvex optimization problems, where in each iteration the nonconvex feasible set is approximated by an inner convex approximation. The latter is defined using an upper bound on the nonconvex constraint functions. Under appropriate conditions, a monotone convergence to a KKT point is established. The scheme is applied to truss topology design (TTD) problems, where the nonconvex constraints are associated with bounds on displacements and stresses. It is shown that the approximate convex problem solved at each inner iteration can be cast as a conic quadratic programming problem, hence large scale TTD problems can be efficiently solved by the proposed method.

Keywords Nonconvex optimization · Successive convex approximations · KKT points · Truss topology design · Displacement and stress constraints

1 Introduction

Consider the following generic optimization problem:

$$(P)\quad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \le 0, \quad i = 1, \dots, m, \\ & x \in \mathbb{R}^n, \end{array}$$

where f, g_i $(i = 1, \dots, m)$ are all continuously differentiable functions over $\mathbb{R}^n$. In addition, we assume that the function f and the last $m - p$ (for $p \le m$) constraint functions

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$g_{p+1}, \ldots, g_m$ are convex over $\mathbb{R}^n$. Therefore, the "nonconvex part" of the problem is due to the nonconvexity of the first p constraint functions $g_1, \ldots, g_p$. The case $p = m$ corresponds to the case when all the constraints are nonconvex. It is also possible to incorporate linear equality constraints in the above formulation without significantly changing the analysis presented in the paper. For the sake of simplicity, we concentrate on the inequality-constrained problem (P).

Suppose that for every $i = 1, \ldots, p$, g_i has a convex upper estimate function, specifically, assume that there exists a set $Y \subseteq \mathbb{R}^r$ (for some positive integer r) and a continuous function $G_i : \mathbb{R}^n \times Y \mapsto \mathbb{R}$ such that

$$g_i(x) \le G_i(x, y) \quad \text{for every} \quad x \in \mathbb{R}^n, y \in Y,$$

where, for a fixed y, the function $G_i(\cdot, y)$ is convex and continuously differentiable. The vector y plays the role of a *parameter vector* and correspondingly Y is called the *admissible parameters set*. In this paper we introduce and analyze a method for solving problem (P) via a sequence of convex problems. The basic idea of the method is that at each iteration we replace each of the nonconvex functions $g_i(x)$ $(i = 1, \ldots, p)$ by the upper convex approximation function $x \mapsto G_i(x, y)$ for some appropriately chosen parameter vector y. Thus, at step $k (k \ge 1)$ of the method it is required to solve a convex problem of the following form:

$$(P_k) \quad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & G_i(x, y_k) \le 0, \quad i = 1, \ldots, p \\ & g_j(x) \le 0, \quad j = p+1, \ldots, m, \\ & x \in \mathbb{R}^n. \end{array}$$

The vector y_k is a fixed parameter vector depending on the solution of the problem (P_{k-1}). The method will be called *sequential parametric convex approximation (SPCA)* method. Specific details on the underlying assumptions and the SPCA method will be given in the next section.

The idea of iteratively replacing nonconvex functions by convex upper estimates is not new. A well known example is the gradient method as applied to an unconstrained minimization problem

$$\min\{f(x) : x \in \mathbb{R}^n\}.$$

Here f is a (possibly) nonconvex function assumed to be continuously differentiable whose gradient satisfies a Lipschitz condition with constant L. The gradient method is usually written as (see e.g., [5])

$$x_k = x_{k-1} - \frac{1}{L} \nabla f(x_{k-1}), \quad k \ge 1.$$

An equivalent presentation of the method is

$$x_k = \underset{x}{\operatorname{argmin}}\, F(x, x_{k-1}),$$

where

$$F(x, y) = f(y) + \nabla f(y)^T (x - y) + \frac{L}{2} \|x - y\|^2 \tag{1.1}$$

is an upper approximation of $f(x)$. The fact that $f(x) \le F(x, y)$ follows from the well known descent lemma (see [5, Proposition A.24]). Thus, at each iteration we replace the function $f(x)$ by its upper approximation $F(x, x_{k-1})$ with the parameter chosen to be the result of the previous iterate ($y = x_{k-1}$). We see that the gradient method is indeed an SPCA-type method.

We also note that convex *underestimates* (in contrast to our convex *overestimates*) are also widely used in algorithms for nonconvex problems. Notably, the αBB method considered in [1,2] is based on a branch-and-bound approach, where a lower bound on the optimal solution is obtained at each node using a generation of a valid convex underestimate via an interval analysis technique. Another technique for solving nonconvex problems using convex underestimates was suggested for factorable programming in [9].

In the context of structural design problems, a specific convex approximation scheme was used in [3] to convexify global *buckling constraints* and gave rise to an SPCA-type method. In this paper we prove the convergence of the general SPCA method to a KKT point under certain mild conditions. The problem analyzed in [3] satisfies these conditions, and hence the convergence of the corresponding SPCA method is proven by the results of this paper. In this paper we apply this method to structural design problems with two other nonconvex constraints: *displacement* and *stress constraints.* These types of constraints are in fact an essential part of any realistic structural design specification; ignoring them may result in severely unstable structures (see the examples in Sect. 5).

The paper layout is as follows. Section 2 describes the details of the SPCA method and provides all the required assumptions. A convergence analysis of the method is provided in Sect. 3. Implementation and analysis of the SPCA method for structural optimization problems with stress and/or displacement constraints is given in Sect. 4. The paper concludes in Sect. 5 with some numerical examples demonstrating the effectiveness of the SPCA method for the aforementioned optimization problems.

2 The sequential parametric convex approximation (SPCA) method

The SPCA method was loosely described in the introduction. What is apparently missing there is the update formula for the parameter sequence y_k. For a full description of the method and the ensuing analysis, we further require the convex upper estimate functions to satisfy the following property:

Property A *For every $i = 1, \ldots, p$ there is a continuous function $\psi_i : \mathbb{R}^n \to \mathbb{R}^r$ such that for any given point $x \in \mathbb{R}^n$, the vector $y := \psi_i(x) \in Y$ satisfies*

$$g_i(x) = G_i(x, y), \tag{2.1}$$

$$\nabla g_i(x) = \nabla_x G_i(x, y). \tag{2.2}$$

Example 2.1 Consider a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ with a Lipschitz gradient with constant L. Then, as explained in the introduction, the function F given in (1.1) is a convex upper approximation of f. In addition, it clearly satisfies the two parts of Property A with $\psi_i(x) = x$.

Property A induces a very natural choice for the parameter vector at each iteration. The SCPA method can now be stated rigorously.

Sequential Parametric Convex Approximation Method (SPCA):

Step 0. Choose an arbitrary starting point x_0 which is feasible to (P), and set $y_1^{(i)} = \psi_i(x_0)(i = 1, \ldots, p)$.
Step k. Compute a solution x_k of the convex problem:

$$(P_k) \quad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & G_i(x, y_k^{(i)}) \le 0, \quad i = 1, \ldots, p, \\ & g_j(x) \le 0, \quad j = p+1, \ldots, m, \\ & x \in \mathbb{R}^n. \end{array}$$

Set $y_{k+1}^{(i)} = \psi_i(x_k)$ for every $i = 1, \ldots, p$, set $k := k + 1$.

The algorithm stops at iteration k if either the KKT necessary optimality conditions are approximately satisfied, i.e.

$$\min_{\lambda \in \mathbb{R}^n} \left\{ ||\nabla_x L(x_k, \lambda)||^2 \mid \lambda \ge 0, \lambda_i = 0 \ \text{ if } \ g_i(x_k) < 0 \right\} \le \varepsilon, \tag{2.3}$$

where $L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i g_i(x)$ is the Lagrangian of the original problem (P), or no improvement in the objective function value $f(\cdot)$ was achieved in the last 10 iterations. The next result shows that the SPCA method produces a sequence of feasible points whose function values are monotonically nonincreasing, namely, the SPCA method is a descent scheme. We will denote the feasible set of (P_k) by X_k and the feasible set of (P) by X. Throughout the paper we will assume that X is nonempty and compact.

Lemma 2.2 *Let $\{x_k\}$ be the sequence generated by the SPCA method. Then for every $k \ge 0$*

i. $X_k \subseteq X$.
ii. $x_k \in X_k \cap X_{k+1}$.
iii. *x_k is a feasible point of (P).*
iv. $f(x_{k+1}) \le f(x_k)$.

Proof

i. Recall that for every $k \ge 0$ and $i = 1, \ldots, p$ we have $g_i(x) \le G_i(x, y_k^{(i)}) \le 0$, implying that every $x \in X_k$ also satisfies all the constraints of (P), i.e. $x \in X$.
ii. $x_k \in X_k$ since it is an optimal solution of (P_k) (and thus also feasible). By the definition of $y_{k+1}^{(i)}$ we have that $G_i(x_k, y_{k+1}^{(i)}) = g_i(x_k) \le 0$ so that $x_k \in X_{k+1}$.
iii. By parts i and ii we have $x_k \in X_k \subseteq X$, so that x_k is a feasible solution of (P).
iv. By part ii of the lemma it follows that x_k is a feasible solution of (P_{k+1}), which means that its objective function value $f(x_k)$ is no less then the optimal value of (P_{k+1}), which is $f(x_{k+1})$. □

A direct consequence of Lemma 2.2 is the following corollary.

Corollary 2.3 *Let $\{x_k\}$ be the sequence generated by the SPCA method. Then the sequence $\{f(x_k)\}$ converges.*

Proof By Lemma 2.2 the sequence $\{f(x_k)\}$ is nonincreasing. In addition, since the feasible set of (P) is compact and nonempty, it follows that the sequence $\{f(x_k)\}$ is bounded below, and thus has a limit. □

3 Convergence analysis and example

In this section we establish a convergence result for the SPCA method. Since the original problem (P) is nonconvex, it is not possible to prove convergence to a global minimum but rather convergence to KKT points under some regularity conditions. The following simple and technical lemma will be used in the convergence proof.

Lemma 3.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable and strictly convex function on a nonempty convex and compact set $S \subseteq \mathbb{R}^n$. Then f is strongly convex on the set S.*

Proof The function $f(x) - f(y) - \langle x - y, \nabla f(y)\rangle$ is continuous and thus attains its minimal value on the compact set S. Denote this value by q. By the definition of strict convexity it follows that $q > 0$. Denote the squared diameter of the set S by $D := \max_{x,y \in S} ||x - y||^2$ and set $c = \frac{q}{D}$. Then,

$$f(x) - f(y) - \langle x - y, \nabla f(y)\rangle \geq q = c \cdot D \geq c||x - y||^2 \quad \text{for every} \quad x, y \in S,$$

proving the strong convexity of f on S. □

Recall that a feasible solution x^* of problem (P) is *regular* if the set of gradients of the active constraints at x^*, $\{\nabla g_i(x^*)\}_{i \in I}$, is linearly independent, where:

$$I = \{i \in [1, m] : g_i(x^*) = 0\}.$$

Proposition 3.2 *Let $\{x_k\}$ be the sequence generated by the SPCA method.*

i. *If the objective function f is strictly convex on the convex hull of the feasible set, then all regular accumulation points of $\{x_k\}$ are KKT points of problem (P).*
ii. *If the sequence $\{x_k\}$ generated by the SPCA method converges to a regular point x^*, then x^* is a KKT point of problem (P).*

Proof

i. By Lemma 3.1 it follows that the strictly convex objective function f is also strongly convex on the convex and compact feasible set X_{k+1}. In particular, there exists $c > 0$ such that for all $k \geq 0$ we have

$$f(x_k) - f(x_{k+1}) \geq \langle x_k - x_{k+1}, \nabla f(x_{k+1})\rangle + c||x_k - x_{k+1}||^2. \tag{3.1}$$

Since x_k is a feasible point of (P_{k+1}) (by Lemma 2.2, ii), and x_{k+1} is its optimum, then from the optimality conditions for (P_{k+1}) (see [5, proposition 2.1.2]), we obtain:

$$\langle x_k - x_{k+1}, \nabla f(x_{k+1})\rangle \geq 0,$$

which combined with (3.1) yields

$$f(x_k) - f(x_{k+1}) \geq c||x_k - x_{k+1}||^2. \tag{3.2}$$

By Corollary 2.3, the sequence $\{f(x_k)\}$ converges and thus the inequality (3.2) implies that

$$||x_k - x_{k+1}|| \to 0. \tag{3.3}$$

Let x^* be an accumulation point of the sequence $\{x_k\}$, we will show that x^* is a KKT point. Since x^* is an accumulation point of $\{x_k\}$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x^*$. Note that by (3.3) it also follows that $x_{n_k-1} \to x^*$, a fact that will be used in the sequel.

Let I be the index set of the active constraints of (P) with respect to x^* and let I_{n_k} be the index set of the active constraints of (P_{n_k}) with respect to x_{n_k}, that is,

$$\begin{aligned} I &= \{i \in [1, m] : g_i(x^*) = 0\}, \\ I_{n_k} &= \{i \in [1, p] : G_i(x_{n_k}, \psi_i(x_{n_k - 1})) = 0\} \cup \{j \in [p+1, m] : g_j(x_{n_k}) = 0\}. \end{aligned}$$

Letting $k \to \infty$ and using (3.3) along with the continuity of g_j, G_i, ψ_i, we have

$$g_j(x_{n_k}) \to g_j(x^*), \quad j = p+1, \ldots, m, \tag{3.4}$$

$$G_i(x_{n_k}, \psi_i(x_{n_k - 1})) \to G_i(x^*, \psi_i(x^*)) = g_i(x^*), \quad i = 1, \ldots, p, \tag{3.5}$$

where the last equality follows from Property A [relation (2.1)]. The limits (3.4) and (3.5) imply that there exists a positive integer K_1 such that

$$I_{n_k} \subseteq I \quad \text{for all} \quad k > K_1. \tag{3.6}$$

Since the functions $G_i, \nabla_x G_i, g_j, \nabla g_j, \psi_i (i = 1, \ldots, p, j = p+1, \ldots, m)$ are all continuous, when $k \to \infty$ we get

$$\begin{aligned} \nabla f(x_{n_k}) &\to \nabla f(x^*), \\ \nabla_x G_i(x_{n_k}, \psi_i(x_{n_k - 1})) &\to \nabla_x G_i(x^*, \psi_i(x^*)) = \nabla g_i(x^*), \quad i = 1, \ldots, p, \\ \nabla g_j(x_{k_j}) &\to \nabla g_j(x^*), \quad j = p+1, \ldots, m, \end{aligned} \tag{3.7}$$

where the equality in (3.7) follows from Property A (Eq. 2.2). In particular, all the gradients of the constraint functions of (P_{n_k}) converge to the corresponding gradients of the constraint functions of (P). This, combined with the inclusion (3.6), implies that there exists a positive integer $K_2 > K_1$ such that x_{n_k} is a regular point of (P_{n_k}) for every $k > K_2$. Therefore, for every $k > K_2$ the KKT conditions are satisfied for problem (P_{n_k}), namely, there exist nonnegative numbers $\mu_1^{n_k}, \ldots, \mu_m^{n_k} \in \mathbb{R}_+$ such that

$$\begin{aligned} \nabla f(x_{n_k}) + \sum_{i=1}^{p} \mu_i^{n_k} \nabla_x G_i(x_{n_k}, \psi_i(x_{n_k - 1})) + \sum_{j=p+1}^{m} \mu_j^{n_k} \nabla g_j(x_{n_k}) = 0, \\ \mu_i^{n_k} G_i(x_{n_k}, \psi_i(x_{n_k - 1})) = 0, \quad i = 1, \ldots, p, \\ \mu_j^{n_k} g_j(x_{n_k}) = 0, \quad j = p+1, \ldots, m. \end{aligned} \tag{3.8}$$

For every $k > K_2$ let $v_k := -\nabla f(x_{n_k})$ and let A_k be the matrix whose columns are the gradients of the constraints corresponding to the index set I, namely, the vectors

$$\{\nabla_x G_i(x_{n_k}, \psi_i(x_{n_k - 1}))\}_{i \in [1,p] \cap I} \cup \{\nabla g_j(x_{n_k})\}_{j \in [p+1,m] \cap I}.$$

Note that by (3.6) and the complementary slackness conditions for (P_{n_k}), we have for all $k > K_2$ that

$$\mu_i^{n_k} = 0 \quad \text{for all } i \notin I.$$

Therefore, Eq. (3.8) reduces to

$$A_k \eta^k = v_k,$$

where $\eta^k = (\mu_i^{n_k})_{i \in I}$ is the vector of all multipliers corresponding to I. If we denote $v := -\nabla f(x^*)$ and define A to be the matrix whose columns are the vectors

$$\{\nabla g_j(x^*)\}_{j \in [1,m] \cap I},$$

then we have that $A_k \to A$ and $v_k \to v$. In addition, A and A_k (for every $k > K_2$) are of full column rank, which immediately implies that

$$\eta^k = (A_k^T A_k)^{-1} A_k^T v_k.$$

We conclude that $\eta^k \to (A^T A)^{-1} A^T v$. Since $\mu_i^{n_k}$ is comprised of the components of η^k and zeros, it follows that it has a limit. Denoting the limit of $\mu_i^{n_k}$ by $\mu_i^* \geq 0$ and taking the limit $k \to \infty$ for the KKT conditions of (P_{n_k}), we obtain

$$\nabla f(x^*) + \sum_{i \in I} \mu_i^* \nabla g_i(x^*) = 0,$$
$$\mu_i^* g_i(x^*) = 0, \quad i \in I.$$

Defining $\mu_i^* = 0$ for every $i \notin I$ we finally conclude that

$$\nabla f(x^*) + \sum_{i=1}^{m} \mu_i^* \nabla g_i(x^*) = 0,$$
$$\mu_i^* g_i(x^*) = 0, \quad i = 1, \ldots, m+1,$$

proving that x^* is a KKT point.

ii. The proof here follows the same line of argument as the derivation in the first part. The only difference is that we do not require the strict convexity assumption in order to establish that both x_k and x_{k-1} converge to the same limit.

□

Remark 3.3 The analysis in the current and previous sections is made under the assumption that the objective and constraint functions are defined over the entire space $\mathbb{R}^n$. The same analysis is applicable to the case when all the functions are defined over an open domain Ω containing X.

Example 3.4 Consider the following nonconvex problem:

$$\begin{aligned} \min\ & f(x) := (x_1 - 2)^2 + (x_2 - 2)^2 \\ \text{s.t.}\ & g(x) := x_1 x_2 \leq 1, \\ & 0.01 \leq x_1, x_2 \leq 100. \end{aligned} \tag{3.9}$$

Obviously the objective function f is strictly convex, the function g is nonconvex and the feasible set of problem (3.9) is compact. For every $\lambda > 0$, define the function

$$G(x, \lambda) := \frac{\lambda}{2} x_1^2 + \frac{1}{2\lambda} x_2^2.$$

In the following lemma we will prove that $G(x, \lambda)$ overestimates $g(x)$ for every $\lambda > 0$ and that Property A is satisfied.

Lemma 3.5 *The function $G(x, \lambda)$ is convex and overestimates the function $g(x)$ for every fixed value $\lambda > 0$, i.e.*

$$g(x) \leq G(x, \lambda), \quad \forall \lambda > 0. \tag{3.10}$$

Let $\psi(x) := \frac{x_2}{x_1}$. Then for a given feasible point $x = (x_1, x_2)$ satisfying $x_1 \neq 0$ and $\lambda := \psi(x)$ it holds that

$$g(x) = G(x, \lambda), \tag{3.11}$$

$$\nabla g(x) = \nabla G(x, \lambda). \tag{3.12}$$

Proof Since $\lambda > 0$, the function G is convex quadratic. The relation (3.10) holds true since

$$G(x, \lambda) - g(x) = \frac{1}{2}\left(\sqrt{\lambda}x_1 - \frac{1}{\sqrt{\lambda}}x_2\right)^2 \geq 0.$$

Now, by substituting $\lambda = \psi(x) = \frac{x_2}{x_1}$ we get:

$$G(x, \lambda) = \frac{\lambda}{2}x_1^2 + \frac{1}{2\lambda}x_2^2 = x_1x_2 = g(x),$$

and thus (3.11) holds true. The gradient of the function G is given by

$$\nabla G(x, \lambda) = \begin{pmatrix} \lambda x_1 \\ x_2/\lambda \end{pmatrix}.$$

Substituting in the above expression $\lambda := \frac{x_2}{x_1}$ yields

$$\nabla G(x, \lambda) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \nabla g(x).$$

□

Since all required assumptions for the SPCA method are satisfied, we can replace the function g by its convex approximation G and solve iteratively the sequence of convex problems:

$$(E_k) \quad \begin{array}{ll} \min & (x_1 - 2)^2 + (x_2 - 2)^2 \\ \text{s.t.} & \frac{\lambda_{k-1}}{2}x_1^2 + \frac{1}{2\lambda_{k-1}}x_2^2 \leq 1, \\ & 0.01 \leq x_1, x_2 \leq 100, \end{array}$$

where the parameter λ_k is updated at each iteration according to the obtained solution, specifically $\lambda_k = \frac{x_2^k}{x_1^k}$ with (x_1^k, x_2^k) being the optimal solution of (E_k).

The optimal solution of the original nonconvex problem (3.9) is attained at the point (1, 1) with a corresponding optimal value of 2. We ran 25 iterations of the SPCA method starting from point (5, 0.02), which is rather far from the optimum and reached the value 2.0015 after 25 iterations. Figure 1 describes the feasible sets and iterates at iterations 1, 2, 3, 4, 5, 25.

We also observed empirically that the method always converges (in this example) to the global optimum as long as the starting point (even an infeasible one) is in the first quadrant.

Remark 3.6 Note that we assume that the initial point x_0 is feasible. This assumption implies the feasibly of the entire sequence of iteration points $\{x_k\}$. Finding an initial point might be an easy or a hard task depending on the specific problem at hand. In Sect. 5 we will describe a method for choosing a feasible point in the context of truss topology design (TTD) problems.

4 Nonconvex truss topology design problems

In this section we will explain how the SPCA method can be applied to solving TTD problems with (nonconvex) stress and displacement constraints. Several methods were proposed in the literature to deal with these types of nonconvex constraints. In [8] a bilevel programming approach is proposed for the minimum-compliance formulation of the TTD problem with

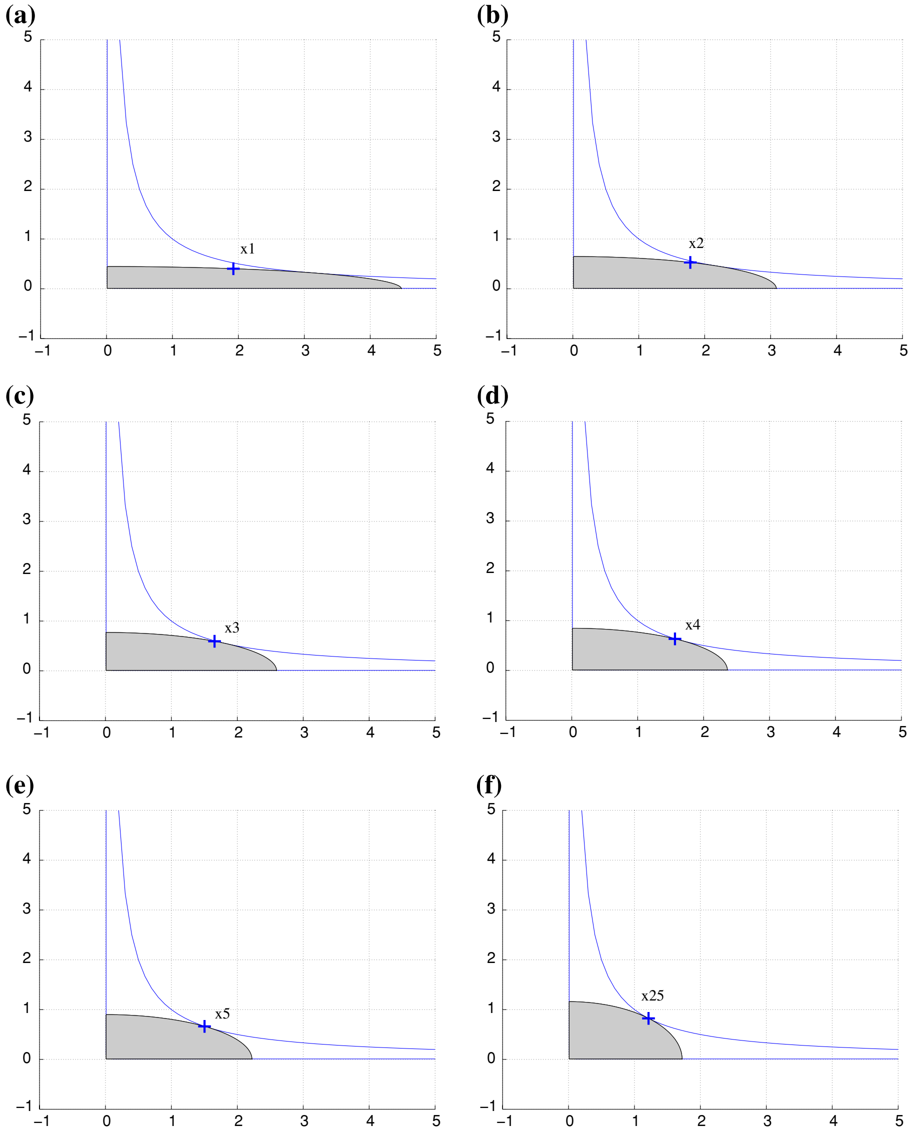

Fig. 1 Iterations $1, \ldots, 5$ and 25 of the SPCA method. The *shaded area* is the feasible set of the convex approximation problem (E_k)

displacement constraints. A relaxation method for the TTD problem with stress constraints has been presented in [7] and was further analyzed in [11]. In their approach, termed *the epsilon-relaxation method*, the stress constraint is relaxed for a bar with small cross-sectional area. As noted in [12], it is very difficult, however, to choose an appropriate value of relaxing parameters to reach the globally optimal solution. An alternative to the epsilon-relaxation methodology was proposed in [6], and is based on the same idea but is different in terms of convergence features. In [12] the same authors proved that the problem can be rewritten as a linear mixed 0–1 problem.

In this section we show that the SPCA method can be applied to TTD problem with displacement and/or stress constraints.

4.1 The basic truss topology design (TTD) problem

A truss is a mechanical construction comprising thin elastic bars linked to each other, such as an electric mast, a railroad bridge, or the Eiffel Tower, see [4]. The goal is to design a truss of a given total weight best able to withstand the given load while satisfying other requirements such as displacement and/or stress constraints or alternatively to find the minimum weight truss satisfying stress and displacement constraints. In the sequel we will concentrate on minimum-weight formulations.

A basic multiload TTD problem is described by the following mathematical formulation which consists of minimizing total material (weight) subject to bounds on the compliance for each of the K loading scenarios:

$$\begin{array}{ll} \min_t & \sum_{i=1}^n t_i \\ \text{s.t.} & f_k^T A(t)^{-1} f_k \leq \gamma \quad k = 1, \dots K, \\ & t_i \geq \epsilon \quad i = 1, \dots, n. \end{array} \tag{4.1}$$

The decision variables $t_i (i = 1, \dots, n)$ are the volumes of the corresponding bars, n is the number of potential bars, f_k is a load vector corresponding to the k-th scenario, γ is a known upper bound on the compliance $f_k^T A(t)^{-1} f_k$ (potential energy stored in the truss), ϵ is a very small positive number and $A(t) = \sum_{i=1}^n t_i b_i b_i^T \in \mathbb{R}^{M \times M}$ is the so-called bar-stiffness matrix. The vectors $b_i \in \mathbb{R}^M$ depend on the coordinates of the nodes and the type of material of the truss. The stiffness matrix is assumed to be a positive definite matrix for all $t > 0$ so that the compliance constraints are well defined.

Problem (4.1) is a convex problem since for every $k = 1, \dots, K$, the compliance function $f_k A(t)^{-1} f_k$ is convex, see [4, Proposition 4.8.1] for details.

4.2 Nonconvex constraints

Consider the function

$$H(t) := |q^T A(t)^{-1} f|, \tag{4.2}$$

where $q, f \in \mathbb{R}^M$ are fixed vectors and $A(t)$ is the bar-stiffness matrix just defined. We will now show that displacement and stress constraints, which are nonconvex, can be formulated using the function H.

4.2.1 Displacement constraints

Recall that the displacement vector corresponding to the k-th scenario is given by $u_k := A(t)^{-1} f_k$. A *displacement constraint* consists of bounding a certain norm of the displacement vector:

$$||A(t)^{-1} f_k|| \leq \rho. \tag{4.3}$$

Depending on the choice of the norm $|| \cdot ||$ we can obtain different representations of the displacement constraint (4.3). For example, in the case of an l_1-norm, the constraint (4.3) can be written in the following form:

$$|e_j^T A(t)^{-1} f_k| \leq \tau_j, \quad j = 1, \dots M,$$
$$\sum_{j=1}^{M} \tau_j \leq \rho, \tag{4.4}$$

where e_j is the j-th unit vector. If we use the l_∞-norm we obtain constraints of the form:

$$|e_j^T A(t)^{-1} f_k| \leq \rho, \quad j = 1, \dots, M. \tag{4.5}$$

Alternatively, we might be interested also to restrict displacements just of certain nodes and not of all nodes. In this case we add only some of the constraints of (4.5). In all the above examples of displacement constraints, the corresponding nonconvex constraints are indeed special cases of the function $H(t)$ given in (4.2).

4.2.2 Stress constraints

Suppose we are interested in restricting stresses in bars by some finite upper bound ν. This can be formulated as follows

$$|\sqrt{E} b_i^T A(t)^{-1} f_k| \leq \nu, \qquad i = 1, \dots, n, \tag{4.6}$$

where E is the Young modulus of the material. The stress constraint function is just the function H with $q = \sqrt{E} b_i$ and $f = f_k$.

To summarize, using the function $H(t)$ defined in (4.2), it is possible to incorporate several types of displacement and stress constraints. All these problems are then modelled by the following nonconvex problems in variables $t \in \mathbb{R}^n$ and $v \in \mathbb{R}^d$:

$$\begin{aligned} \min \ & \textstyle\sum_{i=1}^n t_i \\ \text{s.t.} \ & f_k^T A(t)^{-1} f_k \leq \gamma, \quad k = 1, \dots, K \\ & |q_j^T A(t)^{-1} r_j| \leq \alpha_j v_j + \beta_j, \quad j = 1, \dots, d \\ & Pv + c \leq 0 \\ & t_i \geq \epsilon, \quad i = 1, \dots, n, \end{aligned} \tag{4.7}$$

where $P \in \mathbb{R}^{l \times d}, c \in \mathbb{R}^l, q_1, \dots, q_d, r_1, \dots, r_d \in \mathbb{R}^M, \alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d \in \mathbb{R}$. We will assume the more general setting in which $A(t)$ is given by:

$$A(t) = \sum_{i=1}^{n} t_i B_i B_i^T, \qquad B_i \in \mathbb{R}^{M \times \mu}.$$

The feasible set of (4.7) is compact and we will assume that it is nonempty. To solve (4.7) by the SPCA method, we first develop a convex upper approximation for the generic function $H(t)$ in (4.2) and use it to approximate the nonconvex constraints in (4.7).

4.3 A convex approximation

Consider again the nonconvex function $H(t)$ given in (4.2) and define the function:

$$F_{\lambda,h}(t) = \frac{\lambda}{2} q^T A(t)^{-1} q + \frac{1}{2\lambda} (f + A(t)h)^T A(t)^{-1} (f + A(t)h),$$

where the scalar $\lambda > 0$ and the vector $h \in \mathbb{R}^M$ are fixed parameters. Let us begin by showing that $F_{\lambda,h}(\cdot)$ is convex.

Proposition 4.1 *For a given $\lambda > 0$ and a vector $h \in \mathbb{R}^M$ the function $F_{\lambda,h}(t)$ is convex for any $t > 0$.*

Proof The function $F_{\lambda,h}(t)$ is a linear combination of two functions F_1, F_2 where $F_1 := q^T A(t)^{-1} q$ and $F_2 := (f + A(t)h)^T A(t)^{-1}(f + A(t)h)$. The epigraph of F_1 is given by

$$\{(t, u_1) | u_1 \geq q^T A(t)^{-1} q\},$$

which, by Schur complement, can be equivalently written as

$$\left\{(t, u_1) : \begin{pmatrix} u_1 & q^T \\ q & A(t) \end{pmatrix} \succeq 0\right\}.$$

Similarly, the epigraph of F_2 is given by

$$\left\{(t, u_2) : \begin{pmatrix} u_2 & (f + A(t)h)^T \\ f + A(t)h & A(t) \end{pmatrix} \succeq 0\right\}.$$

Therefore, the epigraphs of F_1 and F_2, being representable by linear matrix inequalities, are convex sets, thus proving that F_1, F_2, and consequently also $F_{\lambda,h}$, are convex functions. □

Next, in Theorem 4.3, we will show that under some conditions on the vector h, $F_{\lambda,h}$ is an upper bound on H. To show this, we will use the following simple lemma.

Lemma 4.2 *Let $\alpha > 0, \beta \geq 0$ be given. Then the optimal solution of the minimization problem*

$$\min_{\lambda \geq 0} \left\{\frac{\lambda}{2}\alpha + \frac{1}{2\lambda}\beta\right\}$$

is $\lambda = \sqrt{\frac{\beta}{\alpha}}$ with a corresponding optimal value $\sqrt{\alpha\beta}$.

Theorem 4.3 *For every scalar $\lambda > 0$ and every vector h such that $q^T h = 0$ it holds that*

$$H(t) \leq F_{\lambda,h}(t) \quad \textit{for every} \quad t > 0.$$

Proof By the fact that $q^T h = 0$ we can write (here $\|\cdot\|$ is the Euclidean l_2 norm)

$$\begin{aligned} H(t) &= |q^T A(t)^{-1} f| = |q^T A(t)^{-1}(f + A(t)h)| \\ &= \left|q^T A(t)^{-1/2} A(t)^{-1/2}(f + A(t)h)\right| \\ &\leq \|A(t)^{-1/2} q\| \cdot \|A(t)^{-1/2}(f + A(t)h)\| \\ &= (q^T A(t)^{-1} q)^{1/2}((f + A(t)h)^T A(t)^{-1}(f + A(t)h))^{1/2}, \end{aligned}$$

where the inequality follows from the Cauchy-Schwartz inequality. Using Lemma 4.2 we have

$$\begin{aligned} H(t) &\leq \underbrace{(q^T A(t)^{-1} q)}_{\alpha}{}^{1/2}\underbrace{((f + A(t)h)^T A(t)^{-1}(f + A(t)h))}_{\beta}{}^{1/2} \\ &= \min_{\tilde{\lambda} \geq 0} \left\{\frac{\tilde{\lambda}}{2} q^T A(t)^{-1} q + \frac{1}{2\tilde{\lambda}}(f + A(t)h)^T A(t)^{-1}(f + A(t)h)\right\} \\ &\leq F_{\lambda,h}(t). \end{aligned}$$

□

Next we show that Property A holds for $H(t)$ and its convex approximation $F_{\lambda,h}(t)$, namely, that for every $t > 0$ there exists a choice of the parameters λ, h for which the function values and gradients of H, $F_{\lambda,h}$ coincide.

Proposition 4.4 *Consider the function H given in (4.2) with $A(t) = \sum_{i=1}^{N} t_i B_i B_i^T$ ($B_i \in \mathbb{R}^{M\times\mu}$) and let $\bar{t} > 0$ with $H(\bar{t}) \neq 0$. Then it holds that*

$$H(t) \leq F_{\bar{\lambda},\bar{h}}(t) \quad \textit{for every} \quad t > 0, \tag{4.8}$$

$$H(\bar{t}) = F_{\bar{\lambda},\bar{h}}(\bar{t}), \tag{4.9}$$

$$\nabla H(\bar{t}) = \nabla F_{\bar{\lambda},\bar{h}}(\bar{t}), \tag{4.10}$$

where

$$\bar{\lambda} = |\theta|,$$
$$\bar{h} = A(\bar{t})^{-1}(\theta q - f).$$

with

$$\theta = \frac{q^T A(\bar{t})^{-1} f}{q^T A(\bar{t})^{-1} q}.$$

Proof By Theorem 4.3, in order to prove (4.8), it is enough to show that $q^T\bar{h} = 0, \bar{\lambda} > 0$. The inequality $\bar{\lambda} > 0$ is satisfied since

$$\bar{\lambda} = |\theta| = \frac{|H(\bar{t})|}{q^T A(\bar{t}) q} > 0.$$

Now,

$$q^T\bar{h} = q^T A(\bar{t})^{-1}(\theta q - f) = \theta q^T A(\bar{t})^{-1} q - q^T A(\bar{t})^{-1} f = 0,$$

thus establishing (4.8). To show (4.9) let us plug the expressions for $\bar{\lambda}$ and $\bar{h}$ in $F_{\bar{\lambda},\bar{h}}(\bar{t})$:

$$F_{\bar{\lambda},\bar{h}}(\bar{t}) = \frac{|\theta|}{2} q^T A(\bar{t})^{-1} q + \frac{1}{2|\theta|}\theta^2 q^T A(\bar{t})^{-1} q = |\theta| q^T A(\bar{t})^{-1} q = |q^T A(\bar{t})^{-1} f| = H(\bar{t}).$$

It remains to check that condition (4.10) is satisfied. Recalling that $A(t) = \sum_{i=1}^{N} t_i B_i B_i^T$, it is not difficult to verify that

$$\left.\frac{\partial F_{\bar{\lambda},\bar{h}}(t)}{\partial t_i}\right|_{t=\bar{t}} = \mathrm{Tr}\left(B_i^T\left(-\frac{\bar{\lambda}}{2}A(\bar{t})^{-1} q q^T A(\bar{t})^{-1} - \frac{1}{2\bar{\lambda}} A(\bar{t})^{-1} f f^T A(\bar{t})^{-1} + \frac{1}{2\bar{\lambda}}\bar{h}\bar{h}^T\right) B_i\right). \tag{4.11}$$

Substituting the expressions for $(\bar{\lambda}, \bar{h})$ into (4.11) we get:

$$\left.\frac{\partial F_{\bar{\lambda},\bar{h}}(t)}{\partial t_i}\right|_{t=\bar{t}} = \mathrm{Tr}\left(B_i^T\left(-\frac{\mathrm{sign}(\theta)}{2}A(\bar{t})^{-1} f q^T A(\bar{t})^{-1} - \frac{\mathrm{sign}(\theta)}{2} A(\bar{t})^{-1} q f^T A(\bar{t})^{-1}\right) B_i\right), \tag{4.12}$$

where the sign of θ equals to the sign of the expression $q^T A(t)^{-1} f$. The function $H(t) := |q^T A(t)^{-1} f|$ is differentiable for all $t > 0$ except for t's in which $|q^T A(t)^{-1} f| = 0$. Since we assumed that $H(\bar{t}) \neq 0$ we have:

$$\left.\frac{\partial H(t)}{\partial t_i}\right|_{t=\bar{t}} = -\text{sign}(\theta)\text{Tr}\left(B_i^T\left(\frac{A(\bar{t})^{-1} f q^T A(\bar{t})^{-1}}{2} + \frac{A(\bar{t})^{-1} q f^T A(\bar{t})^{-1}}{2}\right) B_i\right), \tag{4.13}$$

which is exactly the expression (4.12). □

Remark 4.5 Note that as a consequence of Proposition 4.4, it is required that $H(t) \neq 0$ each time the convex approximation is invoked. In all of our numerical examples we noticed that this assumption is indeed satisfied.

4.4 Implementation of the SPCA algorithm for the TTD problem

The main computational effort in solving the nonconvex TTD probem (4.7) is the solution, at each iteration, of the following convex approximation problem in variables $t_i (i = 1, \ldots, N)$, $v_j (j = 1, \ldots, d)$:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} t_i \\
\text{s.t.} \quad & f_k^T A(t)^{-1} f_k \leq \gamma, \quad k = 1, \ldots, K \\
& \hat{F}_{\lambda_j,h_j}(t) = \frac{\lambda_j}{2} q_j^T A(t)^{-1} q_j + \frac{1}{2\lambda_j}(r_j + A(t)h_j)^T A(t)^{-1} \\
& \qquad \times (r_j + A(t)h_j) \leq \alpha_j v_j + \beta_j, \quad j = 1, \ldots, d \\
& Pv + c \leq 0 \\
& t_i \geq \epsilon, \quad i = 1, \ldots, n.
\end{aligned} \tag{4.14}$$

An important fact, which makes the task of solving the convex approximation problem (4.14) a tractable one, is that it can be cast as a conic quadratic problem (CQ). To see this note first that the convex approximation function $\hat{F}_{\lambda_j,h_j}(t)$ can be written as

$$\hat{F}_{\lambda_j,h_j}(t) = \frac{\lambda_j}{2} q_j^T A(t)^{-1} q_j + \frac{1}{2\lambda_j} r_j^T A(t)^{-1} r_j + \frac{1}{2\lambda_j} h_j^T A(t) h_j + \frac{1}{\lambda_j} r_j^T h_j.$$

Hence, problem (4.14) can be rewritten with the original variables t_j, v_j and additional variables τ_j, θ_j as

$$
\begin{aligned}
&\min \ \textstyle\sum_{i=1}^{n} t_i \\
&\text{s.t.} \quad f_k^T A(t)^{-1} f_k \le \gamma, \quad k = 1, \ldots, K \\
&\qquad q_j^T A(t)^{-1} q_j \le \tau_j, \quad j = 1, \ldots, d \\
&\qquad r_j^T A(t)^{-1} r_j \le \theta_j, \quad j = 1, \ldots, d \\
&\qquad \tfrac{\lambda_j}{2}\tau_j + \tfrac{1}{2\lambda_j}\theta_j + \tfrac{1}{2\lambda_j} h_j^T A(t) h_j + \tfrac{1}{\lambda_j} r_j^T h_j \le \alpha_j v_j + \beta_j, \quad j = 1, \ldots, d \\
&\qquad Pv + c \le 0 \\
&\qquad t_i \ge \epsilon, \quad i = 1, \ldots, n.
\end{aligned} \tag{4.15}
$$

To show that (4.15) can be cast as a CQ problem it suffices to show that a generic constraint

$$
f(t) \equiv q^T A(t)^{-1} q \le \tau \tag{4.16}
$$

has a CQ representation. The latter fact is proved in [4], here we give a shorter self contained proof. We are going to show that $t > 0$ satisfies the inequality (4.16) if and only if there exist vectors $s_i \in \mathbb{R}^{\mu} i = 1, \ldots, n$ that together with t satisfy the system:

$$
\begin{aligned}
\sum_{i=1}^{n} B_i s_i &= q, \\
\sum_{i=1}^{n} \frac{||s_i||^2}{t_i} &\le \tau, \\
t &> 0.
\end{aligned} \tag{4.17}
$$

This system can be further reduced to the following CQ system in variables t_i, s_i and σ_i:

$$
\begin{aligned}
&\sum_{i=1}^{n} B_i s_i = q, \\
&\quad ||s_i||^2 \le t_i \sigma_i, \quad \forall i = 1, \ldots n, \\
&\sum_{i=1}^{n} \sigma_i \le \tau, \\
&\qquad t > 0.
\end{aligned} \tag{4.18}
$$

First, observe that t solves (4.16) if and only there exists $x \in \mathbb{R}^M$ such that

$$
\begin{aligned}
A(t)x = \sum_{i=1}^{n} t_i B_i B_i^T x &= q, \\
q^T x &\le \tau.
\end{aligned} \tag{4.19}
$$

Now assume that (4.19) has a solution (t, x). Define

$$
s_i = t_i B_i^T x \qquad i = 1, \ldots, n. \tag{4.20}
$$

From (4.20) it follows immediately that:

$$\sum_{i=1}^{n} \frac{||s_i||^2}{t_i} = \sum_{i=1}^{n} \underbrace{t_i x^T B_i B_i^T}_{q^T} x = q^T x \leq \tau$$

and

$$\sum_{i=1}^{n} B_i s_i = \sum_{i=1}^{n} t_i B_i B_i^T x = q.$$

Hence we proved that if (t, x) solves (4.19), then there exist vectors $s_1, \ldots, s_n$ such that $(t, s_1, \ldots, s_n)$ solves (4.17).
Conversely, suppose that (4.17) has a solution $(t, s_1, \ldots, s_n)$, then

$$\min_s \left\{ \sum_{i=1}^{n} \frac{||s_i||^2}{t_i} : \sum_{i=1}^{n} B_i s_i = q \right\} \leq \tau, \tag{4.21}$$

and the optimal solution, which we denote by $\bar{s}$, surely satisfies (4.17). The KKT optimality conditions for this problem imply that there exists a vector of multipliers $y \in \mathbb{R}^M$ for which

$$\nabla_{s_i} \left[\sum_{i=1}^{n} \frac{||\bar{s}_i||^2}{t_i} + \left(q - \sum_{i=1}^{n} B_i \bar{s}_i \right)^T y \right]_{s_i = \bar{s}_i} = 0, \quad i = 1, \ldots, n$$

i.e.

$$2\frac{\bar{s}_i}{t_i} - B_i^T y = 0, \qquad i = 1, \ldots, n.$$

Hence, the vector $x = 2y$ satisfies

$$\bar{s}_i = t_i B_i^T x, \qquad i = 1, \ldots, n.$$

Substituting $\bar{s}_i$ in the objective function in (4.21) we get

$$q^T x \leq \tau$$

where

$$q = \sum_{i=1}^{n} B_i \bar{s}_i = \sum_{i=1}^{n} t_i B_i B_i^T x,$$

implying that (t, x) solves (4.19).
Summing up, the CQ problem solved at each iteration is the following:

$$\begin{aligned}
\text{minimize} \quad & w \\
\text{s.t.} \quad & s_{ij}^2 \leq t_i \sigma_{ij}, \qquad i = 1, \ldots, n, \ j = 1, \ldots, K, \\
& p_{ij}^2 \leq t_i \hat{p}_{ij}, \qquad i = 1, \ldots, n, \ j = 1, \ldots, d, \\
& l_{ij}^2 \leq t_i \hat{l}_{ij}, \qquad i = 1, \ldots, n, \ j = 1, \ldots, d, \\
& \sum_{i=1}^{n} \sigma_{ij} \leq \gamma, \qquad j = 1, \ldots, K,
\end{aligned}$$

$$
\begin{aligned}
&\sum_{i=1}^{n} \hat{p}_{ij} \le \tau_j, \qquad j = 1, \ldots, d,\\
&\sum_{i=1}^{n} \hat{l}_{ij} \le \theta_j, \qquad j = 1, \ldots, d,\\
&\frac{\lambda_j}{2}\tau_j + \frac{\theta_j}{2\lambda_j} + \frac{1}{2\lambda_j}\sum_{i=1}^{n}(h_j^T B_i B_i^T h_j)t_i + \frac{1}{\lambda_j} r_j^T h_j \le \alpha_j v_j + \beta_j,\ \ j = 1, \ldots, d,\\
&\sum_{i=1}^{n} s_{ij} b_i = f_j, \qquad j = 1, \ldots, K,\\
&\sum_{i=1}^{n} p_{ij} b_i = q_j, \qquad j = 1, \ldots, d,\\
&\sum_{i=1}^{n} l_{ij} b_i = r_j, \qquad j = 1, \ldots, d,\\
&Pv + c \le 0,\\
&\sum_{i=1}^{n} t_i \le w,\\
&t_i \ge \epsilon, \qquad\qquad i = 1, \ldots, n
\end{aligned}
\tag{4.22}
$$

with variables $t_i, s_{ij}, \sigma_{ij}, p_{ij}, \hat{p}_{ij}, l_{ij}, \hat{l}_{ij}, v_j, \tau_j$ and θ_j.

5 Computational results

In this section we describe several numerical experiments showing the effectiveness of the SPCA method as applied to TTD problems with displacement constraints. The convex subproblems were cast as conic quadratic problems (4.22), and solved by MOSEK software package. In all the TTD examples here, we solve a minimum weight (volume) problem subject to compliance, displacement and/or stress constraints. To find an initial feasible point, all the bar weights were chosen to be equal to some constant. We then enlarged this constant until all constraints were satisfied.

Example 5.1 Consider the single load TTD problem with two vertical external forces described in Fig. 2 with the following ground structure: 18 free nodes, 2 fixed nodes, 117 potential bars. The forces are marked by the two arrows and the fixed nodes are marked by black squares. The logarithm of the total material volume, $p^* = \log w$, for the solution of the convex TTD problem without displacement constraints, that is, problem (4.1) was equal to $p^* = 7.47$. The optimal truss is described by solid lines and the displaced truss is given by by the dashed lines. The maximal displacement in the optimal structure is 0.66. We added (nonconvex) displacement constraints to the two nodes on which the forces act; these were indeed the nodes in which the displacement was the largest (as can be clearly seen in Fig. 2). We limited the displacement at each of these nodes to be at most 0.1. In Fig. 3 we can see that the solution of the TTD problem with the additional displacement constraints is much more stable. The cost is that the total material volume of the new optimal truss increased ($p^* = 9.37$). We obtained this solution in five iterations of the SPCA method after which no significant reduction of the total material volume occurred.

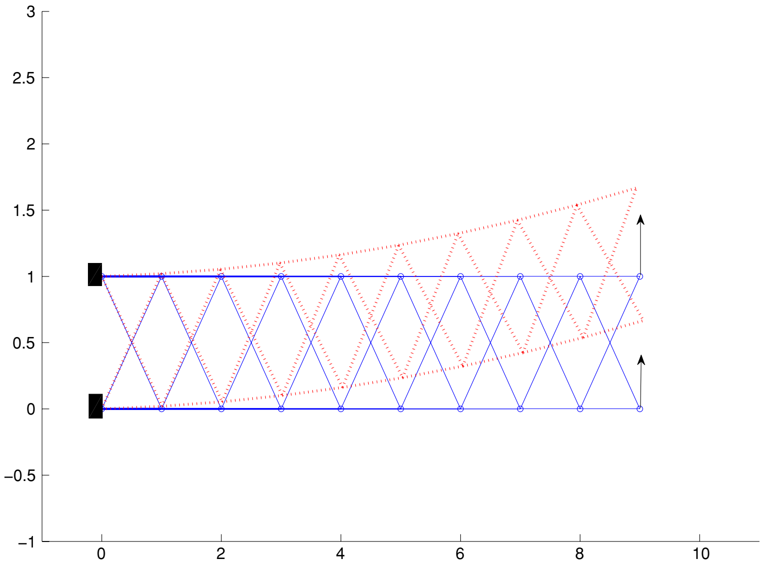

Fig. 2 Maximal displacement equals to 0.66 and $\log(w) = 7.47$

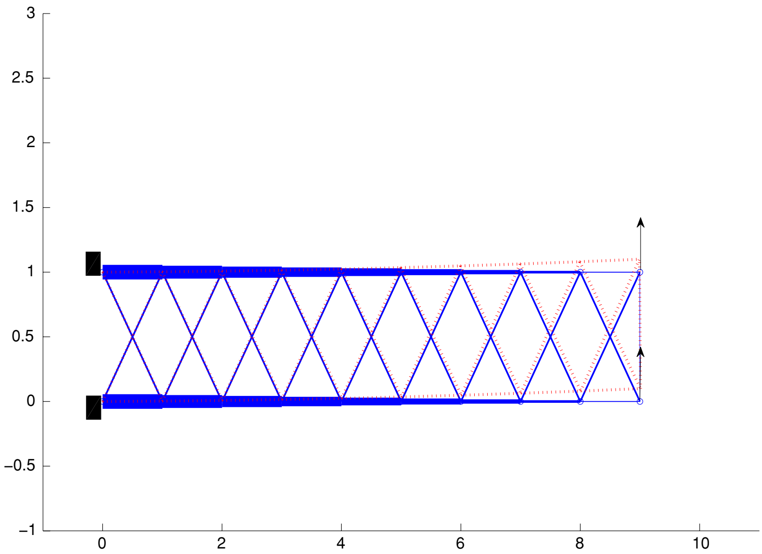

Fig. 3 Maximal displacement equals to 0.1 and $\log(w) = 9.37$

Example 5.2 This example also deals with the single load scenario problem with three vertical forces. The ground structure consists of 40 free nodes, 2 fixed nodes and 559 potential bars. In Fig. 4 we can see the optimal truss and its displacement for the basic convex TTD problem. In Fig. 5 the optimal design obtained after adding displacement constraints is shown. As in the previous example, we restricted the displacement at the three nodes in which the forces act to be no more than 0.1. Here the solution was obtained after 10 iterations of the

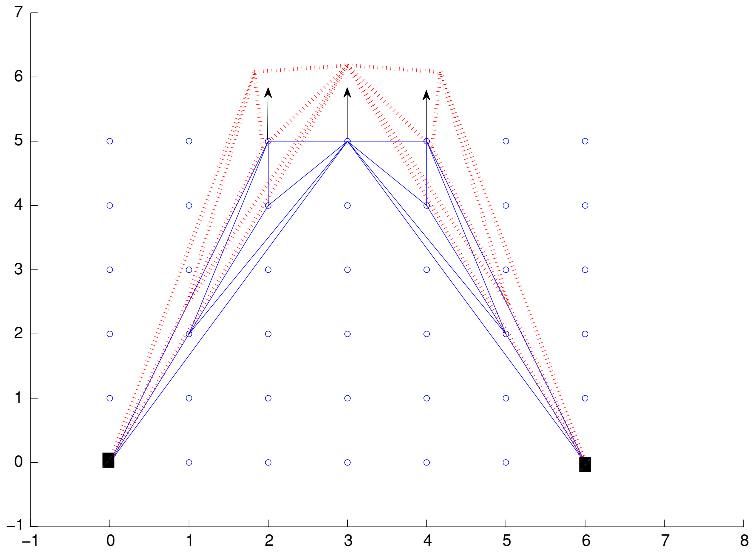

Fig. 4 Maximal displacement equals to 1.18 and $\log(w) = 1.89$

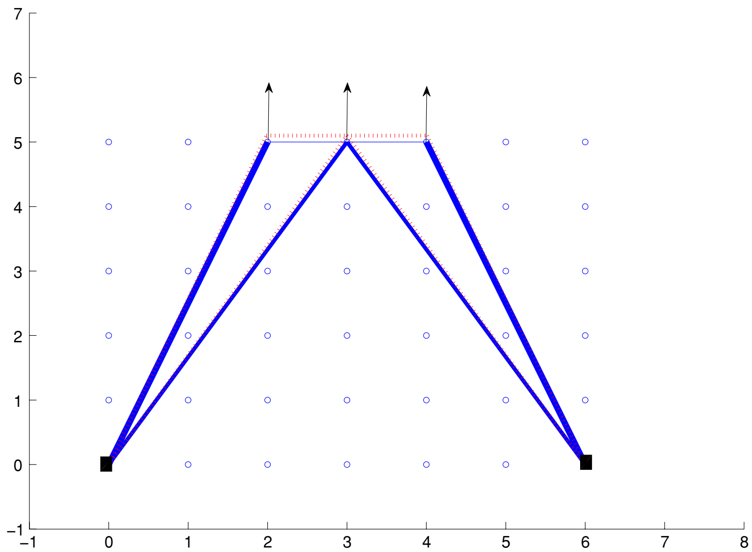

Fig. 5 Maximal displacement equals to 0.1 and $\log(w) = 4.31$

SPCA method. The value of the objective function p^* increased from 1.89 to 4.31 and the maximal displacement was reduced from 1.18 to 0.1.

Example 5.3 In the third example we solved a multiload TTD problem with the same ground structure as in the previous example and with three load scenarios. In each of the three scenarios, one vertical force acts on the truss (colored arrows in Fig. 6). In Fig. 6 we can see the optimal truss obtained by solving the basic convex TTD problem. The displacement caused by each scenario is shown in Fig. 7. We then limited the maximal displacement in the three

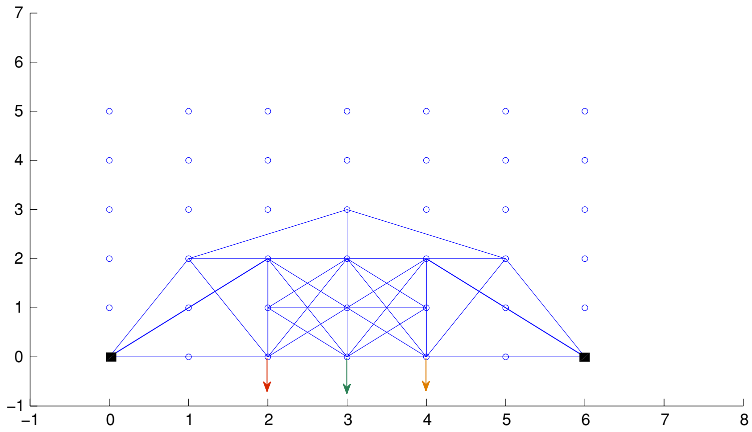

Fig. 6 Topology design of the truss with three load scenarios

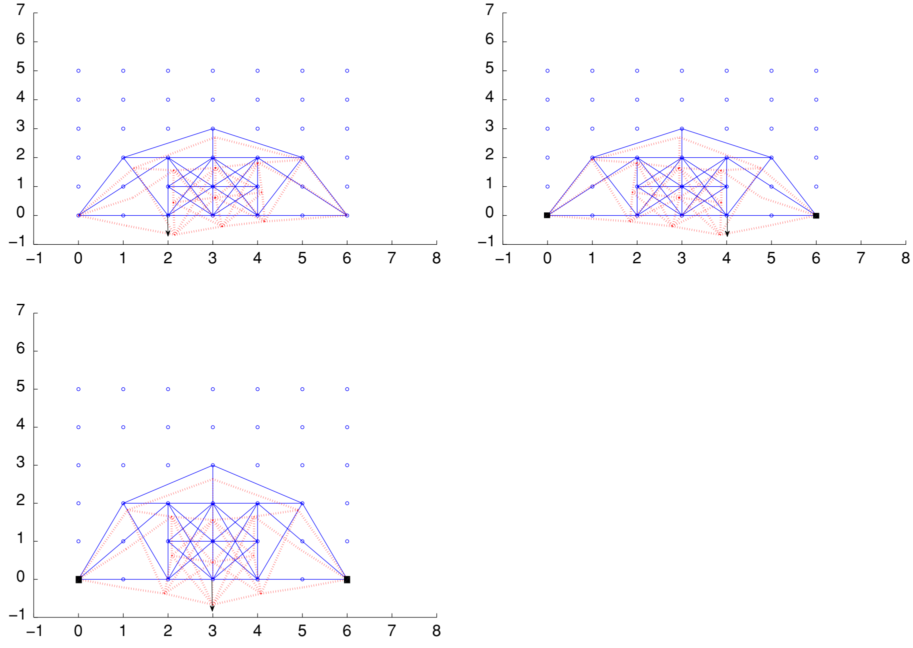

Fig. 7 Maximal displacement of the truss over all three scenarios equals to 0.67 and $\log(w) = 2.21$

nodes on which the force are activated to be no more than 0.1. In the last three images of Fig. 8, we see the truss design and the corresponding displacement for each of the three scenarios. This solution was obtained after two iterations of the SPCA method.

The CPU time of each experiment was <1 min.

Example 5.4 The example deals with a three dimensional problem with 25 potential bars, two load scenarios and 86 nonconvex constraints (36 displacement and 50 stress constraints). The data for this example is based on problem 16 in [10]. The lower bound for all cross

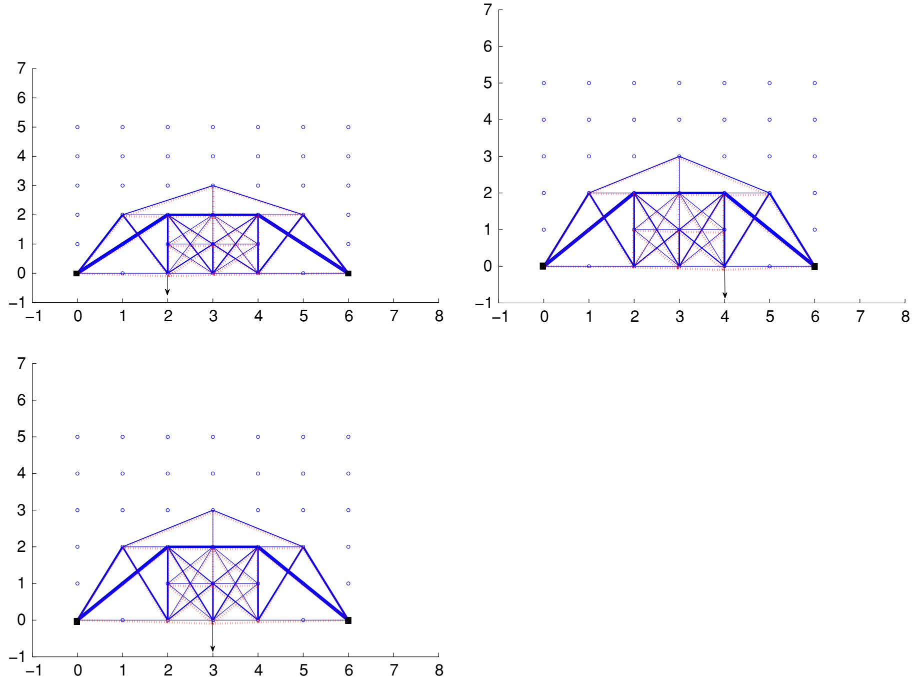

Fig. 8 SPCA solution: maximal displacement of the truss over all three scenarios equals to 0.1 and $\log(w) = 4.1$

Table 1 Lengths of bars

i		Comments
l_i	75	$i = 1, 10, 11, 12, 13$
l_i	$\frac{25}{2}\sqrt{109}$	$i = 2, 3, 4, 5$
l_i	$\frac{25}{2}\sqrt{73}$	$i = 6, 7, 8, 9$
l_i	$25\sqrt{\frac{105}{2}}$	$i = 14, \ldots, 21$
l_i	$25\sqrt{\frac{57}{2}}$	$i = 22, 23, 24, 25$

sections equals to 0.01, the bound for all displacements is 0.35 and the bound for all stresses is 2000. The density of material is 0.1 and Young modulus equals to 10^7. The lengths of bars and loads are given in Tables 1 and 2, respectively. The numbering of the bars and the nodes can be seen in Fig. 9 (the numbers in parenthesis correspond to bars and the bold numbers correspond to nodes).

Results. At the starting point the weight was 3000 (equally distributed between 25 bars). It was reduced after 200 iterations (CPU: 6 min) to 1010.9. From there, no significant progress in the objective value was observed and the algorithm was stopped. The stopping rule (2.3) based on the KKT conditions was valid with $\varepsilon = 10^{-4}$. The final structure is depicted in Fig. 9.

Table 2 Load conditions

j		Comments
f_{j1}	1000	$j = 1$
f_{j1}	10000	$j = 2, 5$
f_{j1}	−5000	$j = 3, 6$
f_{j1}	500	$j = 7, 16$
f_{j1}	0	Otherwise
f_{j2}	20000	$j = 2$
f_{j2}	0.1	$j = 1, 4$
f_{j2}	−5000	$j = 3, 6$
f_{j2}	−20000	$j = 5$
f_{j2}	0	Otherwise

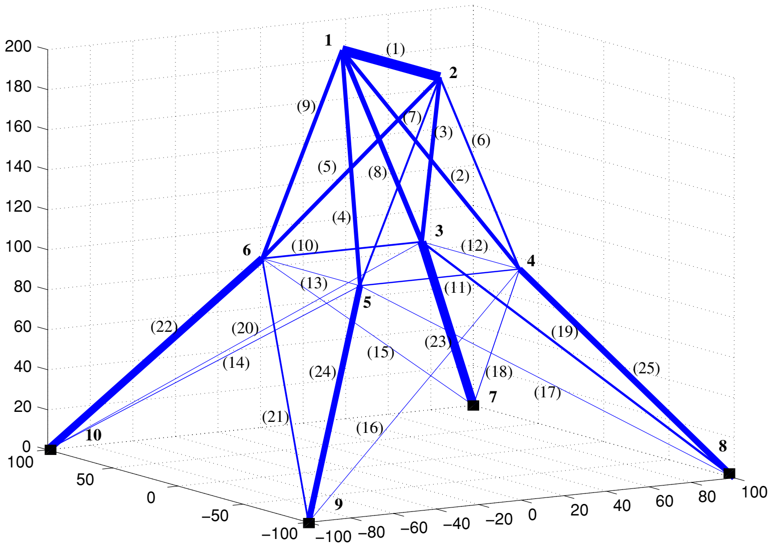

Fig. 9 The truss topology in the Example 5.4

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